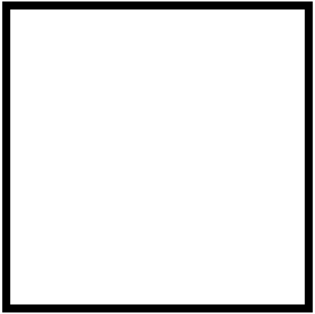
Cutting squares into equal-area triangles

Jim Fowler

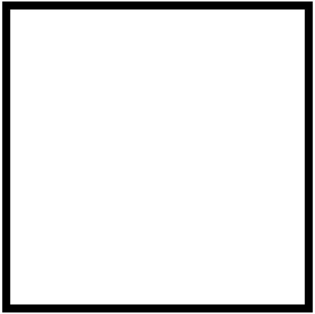
A talk for $\sqrt{\pi}$

Can you cut a square



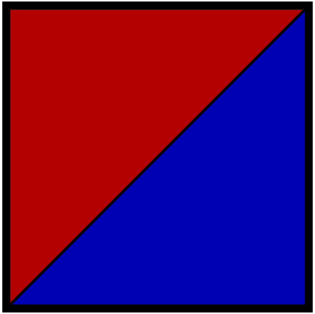
into equal-area triangles?

Can you cut a square



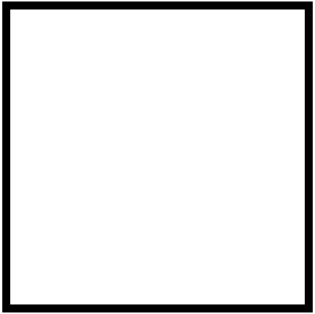
into 2 equal-area triangles?

Can you cut a square



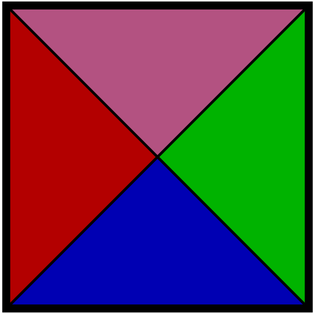
into 2 equal-area triangles? **Yes.**

Can you cut a square



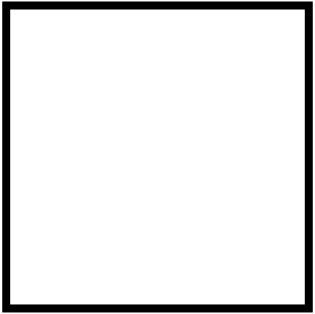
into 4 equal-area triangles?

Can you cut a square



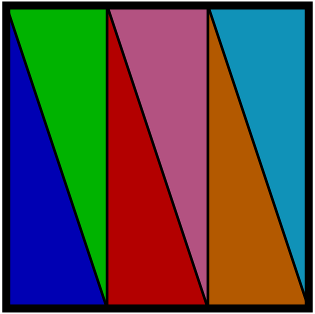
into 4 equal-area triangles? **Yes.**

Can you cut a square



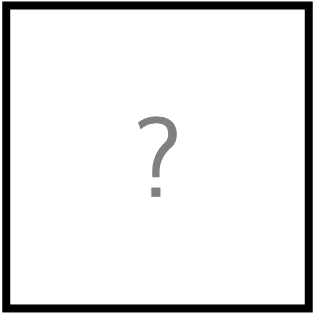
into 6 equal-area triangles?

Can you cut a square



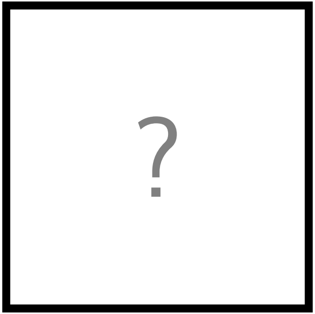
into 6 equal-area triangles? **Yes.**

Can you cut a square



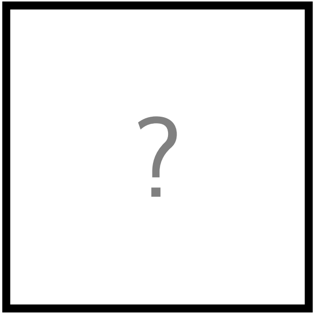
into **3** equal-area triangles?

Can you cut a square



into **5** equal-area triangles?

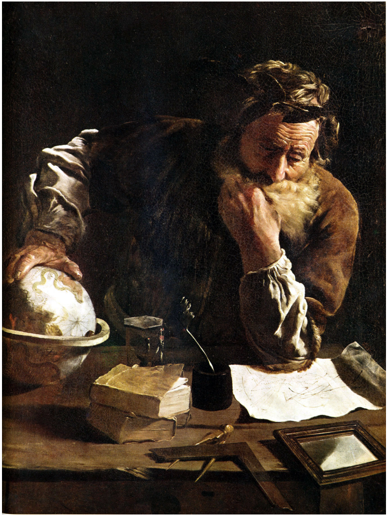
Can you cut a square



into an odd number of equal-area triangles?

Question

Can a square be divided into an odd number of triangles each having the same area?



This question could have been asked two thousand years ago!


How do we split
a square sandwich
among 5 friends?

Question

Can a square be divided into an odd number of triangles each having the same area?

Question

Can a square be divided into an odd number of triangles each having the same area?

Answer

No. It cannot be done.

Question

Can a square be divided into an odd number of triangles each having the same area?

Answer

No. It cannot be done.

Why not. . . ?

The Proof

Two ingredients:

Sperner's Lemma

and a

2-adic valuation.

The Proof

Two ingredients:

Sperner's Lemma

and a

2-adic valuation.

Some Notation

$$a \equiv b \pmod{2}$$

means a and b are **both even** or **both odd**,

Some Notation

$$a \equiv b \pmod 2$$

means a and b are **both even** or **both odd**,
means $a - b$ is divisible by two.

Some Notation

$$a \equiv b \pmod{2}$$

means a and b are **both even** or **both odd**,
means $a - b$ is divisible by two.

"a is congruent to b, modulo two."

Some Notation

$$a \equiv b \pmod 2$$

means a and b are **both even** or **both odd**,
means $a - b$ is divisible by two.

"a is congruent to b, modulo two."

$1 \equiv \ 3 \pmod 2$,

Some Notation

$$a \equiv b \pmod{2}$$

means a and b are **both even** or **both odd**,
means $a - b$ is divisible by two.

"a is congruent to b, modulo two."

$1 \equiv \ 3 \pmod{2}$, but $2 \not\equiv 5 \pmod{2}$.

Some Notation

$$a \equiv b \pmod{2}$$

means a and b are **both even** or **both odd**,
means $a - b$ is divisible by two.

"a is congruent to b, modulo two."

$1 \equiv \; 3 \pmod{2}$, but $2 \not\equiv 5 \pmod{2}$.
$0 \equiv 16 \pmod{2}$,

Some Notation

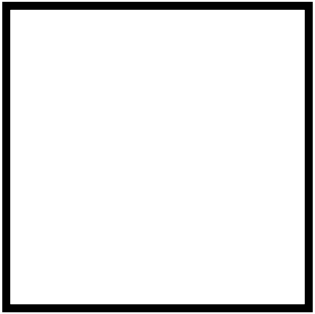
$$a \equiv b \pmod 2$$

means a and b are **both even** or **both odd**,
means $a - b$ is divisible by two.

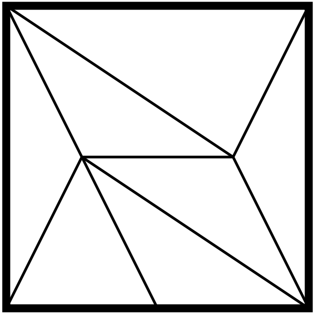
"a is congruent to b, modulo two."

$1 \equiv \ 3 \pmod 2$, but $2 \not\equiv 5 \pmod 2$.
$0 \equiv 16 \pmod 2$, but $1 \not\equiv 2 \pmod 2$.

Sperner's Lemma

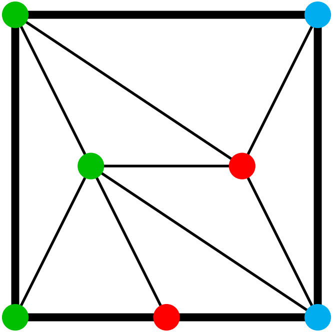


Sperner's Lemma



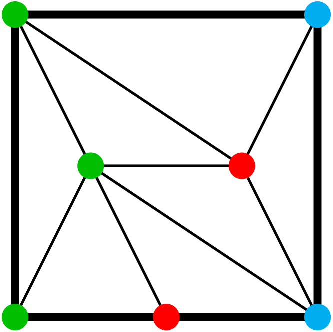
Triangulate.

Sperner's Lemma



Triangulate. **Color** vertices **A**, **B**, or **C**.

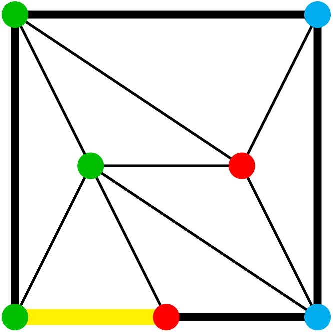
Sperner's Lemma



Triangulate. **Color** vertices **A**, **B**, or **C**.

$$\frac{\textbf{AB}\text{ edges}}{\text{on perimeter}} \equiv \frac{\textbf{ABC}}{\text{triangles}} \pmod 2$$

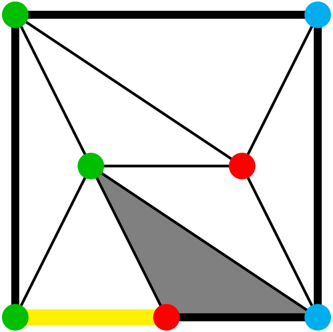
Sperner's Lemma



Triangulate. **Color** vertices **A**, **B**, or **C**.

$$\frac{\textbf{AB}\text{ edges}}{\text{on perimeter}} \equiv \frac{\textbf{ABC}}{\text{triangles}} \pmod 2$$

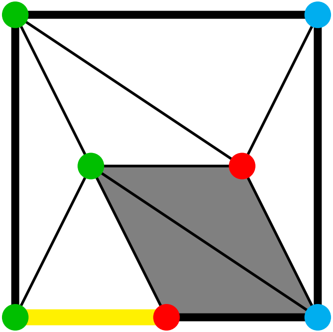
Sperner's Lemma



Triangulate. **Color** vertices **A**, **B**, or **C**.

$$\frac{\textbf{AB}\text{ edges}}{\text{on perimeter}} \equiv \frac{\textbf{ABC}}{\text{triangles}} \pmod 2$$

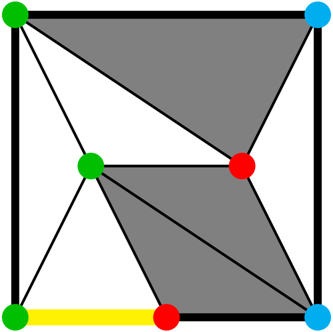
Sperner's Lemma



Triangulate. **Color** vertices **A**, **B**, or **C**.

$$\frac{\textbf{AB}\text{ edges}}{\text{on perimeter}} \equiv \frac{\textbf{ABC}}{\text{triangles}} \pmod 2$$

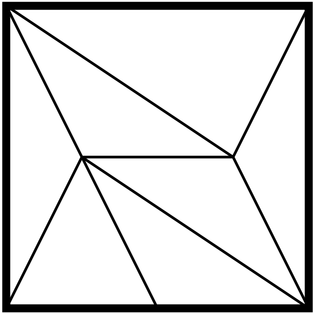
Sperner's Lemma



Triangulate. **Color** vertices **A**, **B**, or **C**.

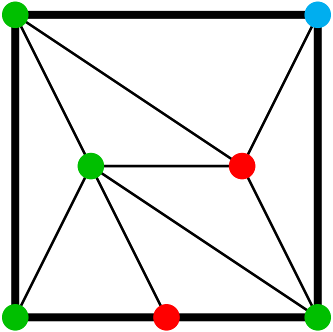
$$\frac{\textbf{AB}\text{ edges}}{\text{on perimeter}} \equiv \frac{\textbf{ABC}}{\text{triangles}} \pmod 2$$

It doesn't matter how we color the vertices.



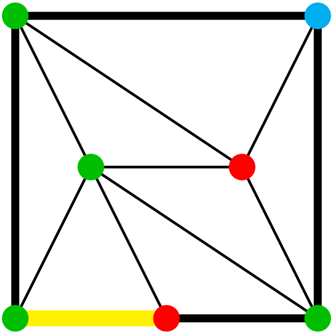
$$\frac{\textbf{AB}\text{ edges}}{\text{on perimeter}} \equiv \frac{\textbf{ABC}}{\text{triangles}} \pmod 2$$

It doesn't matter how we color the vertices.



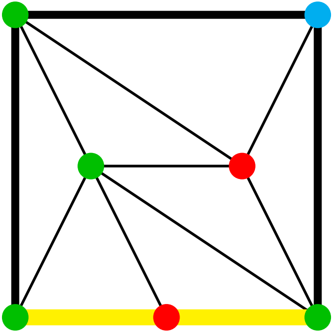
$$\frac{\text{AB edges}}{\text{on perimeter}} \equiv \frac{\text{ABC}}{\text{triangles}} \pmod 2$$

It doesn't matter how we color the vertices.



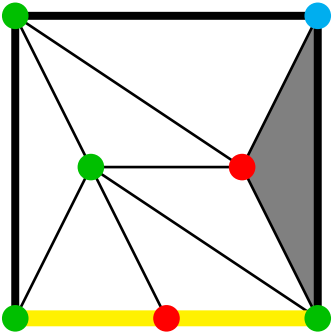
AB edges on perimeter $\equiv$ ABC triangles (mod 2)

It doesn't matter how we color the vertices.



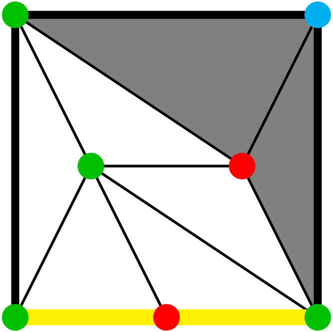
AB edges on perimeter $\equiv$ **ABC** triangles (mod 2)

It doesn't matter how we color the vertices.



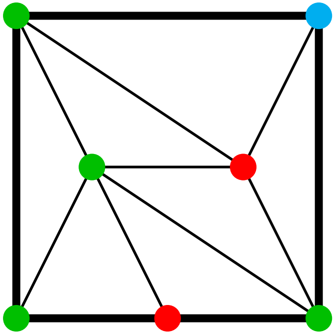
AB edges on perimeter ≡ ABC triangles (mod 2)

It doesn't matter how we color the vertices.



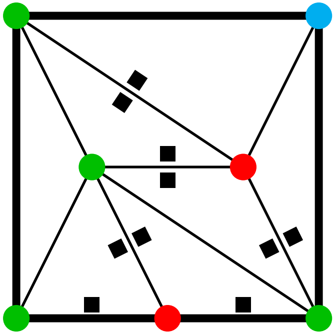
AB edges on perimeter $\equiv$ **ABC** triangles (mod 2)

A Proof of Sperner's Lemma



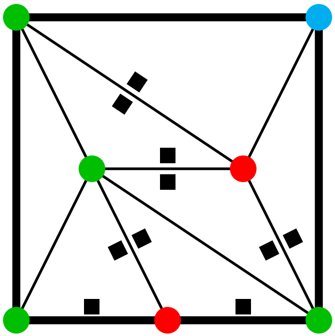
A Proof of Sperner's Lemma

On the inside of the square, on each side of **AB** edges, place a pebble.



A Proof of Sperner's Lemma

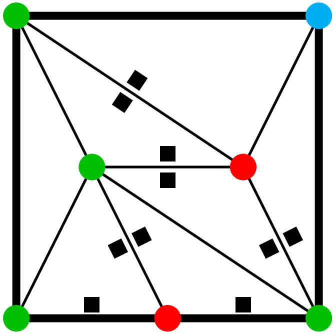
On the inside of the square, on each side of **AB** edges, place a pebble.



Count the pebbles

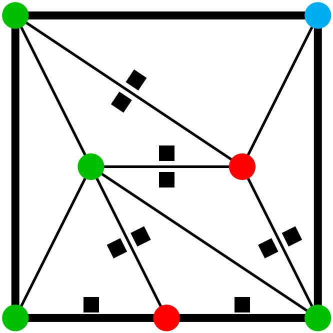
A Proof of Sperner's Lemma

On the inside of the square, on each side of **AB** edges, place a pebble.

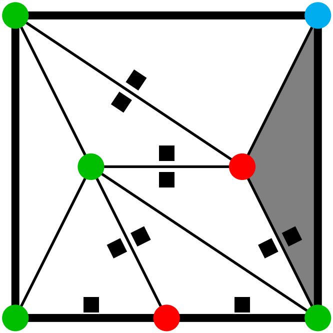


Count the pebbles—in two different ways.

A Proof of Sperner's Lemma

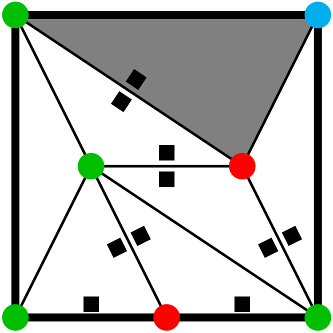


A Proof of Sperner's Lemma



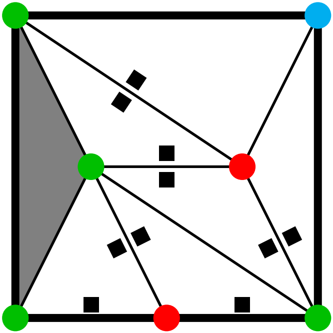
Each ABC triangle gives one pebble.

A Proof of Sperner's Lemma



Each ABC triangle gives one pebble.

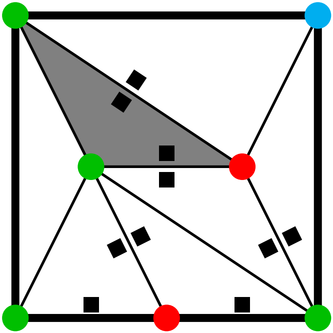
A Proof of Sperner's Lemma



Each **ABC** triangle gives one pebble.

Other triangles give zero or two pebbles.

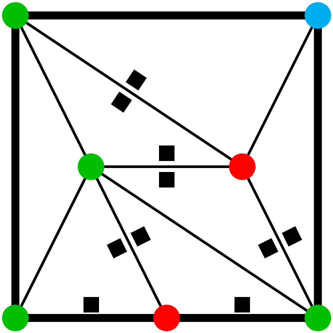
A Proof of Sperner's Lemma



Each **ABC** triangle gives one pebble.

Other triangles give zero or two pebbles.

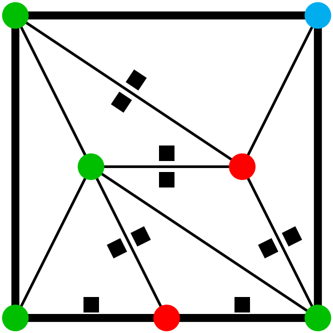
A Proof of Sperner's Lemma



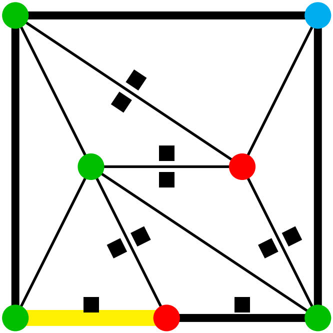
Each **ABC** triangle gives one pebble.
Other triangles give zero or two pebbles.

$$\text{pebbles} \equiv \frac{\textbf{ABC}}{\text{triangles}} \pmod 2$$

A Proof of Sperner's Lemma

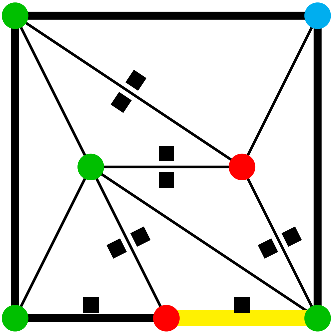


A Proof of Sperner's Lemma



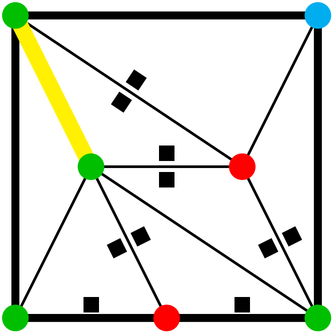
Each **AB** edge on the perimeter gives one pebble.

A Proof of Sperner's Lemma



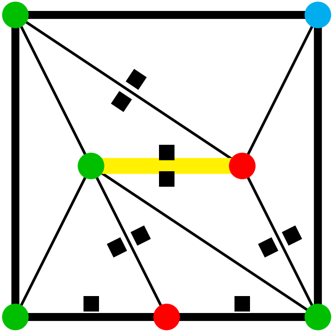
Each AB edge on the perimeter gives one pebble.

A Proof of Sperner's Lemma



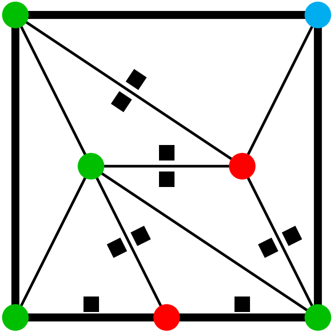
Each **AB** edge on the perimeter gives one pebble. Other edges give zero or two pebbles.

A Proof of Sperner's Lemma



Each **AB** edge on the perimeter gives one pebble. Other edges give zero or two pebbles.

A Proof of Sperner's Lemma



Each **AB** edge on the perimeter gives one pebble. Other edges give zero or two pebbles.

$$\text{pebbles} \equiv \begin{array}{c}\textbf{AB}\text{ edges}\\ \text{on perimeter}\end{array} \pmod 2$$

Proof of Sperner's Lemma

$$\underset{\text{triangles}}{\textbf{ABC}} \equiv \text{pebbles} \pmod 2$$

Proof of Sperner's Lemma

$$\underset{\text{triangles}}{\textbf{ABC}} \equiv \text{pebbles} \pmod 2 \text{ and}$$

$$\underset{\text{on perimeter}}{\textbf{AB}\text{ edges}} \equiv \text{pebbles} \pmod 2$$

Proof of Sperner's Lemma

$$\begin{array}{c}\textbf{ABC}\\ \text{triangles}\end{array} \equiv \text{pebbles} \pmod 2 \text{ and}$$

$$\begin{array}{c}\textbf{AB}\text{ edges}\\ \text{on perimeter}\end{array} \equiv \text{pebbles} \pmod 2 \text{ so}$$

$$\begin{array}{c}\textbf{AB}\text{ edges}\\ \text{on perimeter}\end{array} \equiv \begin{array}{c}\textbf{ABC}\\ \text{triangles}\end{array} \pmod 2.$$

Sperner's Lemma

No matter how you triangulate,
or how you color the vertices,

the number of perimeter **AB** edges, and
the number of **ABC** triangles

are both odd or both even.

Applying Sperner's Lemma

An odd number of perimeter **AB** edges $\Rightarrow$
An odd number of **ABC** triangles.

Applying Sperner's Lemma

An odd number of perimeter **AB** edges $\Rightarrow$
An odd number of **ABC** triangles.

An odd number of perimeter **AB** edges $\Rightarrow$
there exists an **ABC** triangle!

The Proof

Two ingredients:

Sperner's Lemma

and a

2-adic valuation.

The Proof

Two ingredients:

Sperner's Lemma

and a

2-adic valuation.

2-adic valuation

In addition to Sperner's lemma,
we need a **2-adic valuation**.

2-adic valuation

In addition to Sperner's lemma,
we need a **2-adic valuation**.

Before talking about *2-adic* valuations, let's talk about valuations.

Valuations

Absolute value is an example of a valuation.

Valuations

Absolute value is an example of a valuation.

A **valuation** measures how big a number is.

Valuations

Absolute value is an example of a valuation.

A **valuation** measures how big a number is.

A **valuation** is a function $|\cdot| : \mathbb{R} \to \mathbb{R}$

- $|x| \geq 0$ for all $x \in \mathbb{R}$.

Valuations

Absolute value is an example of a valuation.

A **valuation** measures how big a number is.

A **valuation** is a function $|\cdot| : \mathbb{R} \to \mathbb{R}$

- $|x| \geq 0$ for all $x \in \mathbb{R}$.
- $|x| = 0$ if and only if $x = 0$.

Valuations

Absolute value is an example of a valuation.

A **valuation** measures how big a number is.

A **valuation** is a function $|\cdot| : \mathbb{R} \to \mathbb{R}$

- $|x| \geq 0$ for all $x \in \mathbb{R}$.
- $|x| = 0$ if and only if $x = 0$.
- $|x \cdot y| = |x| \cdot |y|$.

Valuations

Absolute value is an example of a valuation.

A **valuation** measures how big a number is.

A **valuation** is a function $|\cdot| : \mathbb{R} \to \mathbb{R}$

- $|x| \geq 0$ for all $x \in \mathbb{R}$.
- $|x| = 0$ if and only if $x = 0$.
- $|x \cdot y| = |x| \cdot |y|$.
- $|x + y| \leq |x| + |y|$.

Valuations

Absolute value is an example of a valuation.

A **valuation** measures how big a number is.

A **valuation** is a function $|\cdot| : \mathbb{R} \to \mathbb{R}$

- $|x| \geq 0$ for all $x \in \mathbb{R}$.
- $|x| = 0$ if and only if $x = 0$.
- $|x \cdot y| = |x| \cdot |y|$.
- $|x + y| \leq |x| + |y|$.

Are there other valuations?

2-adic valuation

A rational number $x = p/q$ can be written as

$$x = 2^n \cdot \frac{a}{b} \text{ for odd numbers } a \text{ and } b.$$

2-adic valuation

A rational number $x = p/q$ can be written as

$$x = 2^n \cdot \frac{a}{b} \text{ for odd numbers } a \text{ and } b.$$

The **2-adic valuation** of x is

$$|x|_2 = (1/2)^n.$$

2-adic valuation

A rational number $x = p/q$ can be written as

$$x = 2^n \cdot \frac{a}{b} \text{ for odd numbers } a \text{ and } b.$$

The **2-adic valuation** of x is

$$|x|_2 = (1/2)^n.$$

This is a different way of measuring the size of a number

2-adic valuation

A rational number $x = p/q$ can be written as

$$x = 2^n \cdot \frac{a}{b} \text{ for odd numbers } a \text{ and } b.$$

The **2-adic valuation** of x is

$$|x|_2 = (1/2)^n.$$

This is a different way of measuring the size of a number—numbers that are divisible by powers of two are small.

2-adic valuation

a and b odd, $x = 2^n \cdot \frac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

2-adic valuation

a and b odd, $x = 2^n \cdot \dfrac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0?$$

2-adic valuation

a and b odd, $x = 2^n \cdot \frac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$

$$|1|_2 =$$

2-adic valuation

a and b odd, $x = 2^n \cdot \frac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$

2-adic valuation

a and b odd, $x = 2^n \cdot \dfrac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 =$$

2-adic valuation

a and b odd, $x = 2^n \cdot \dfrac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$

2-adic valuation

a and b odd, $x = 2^n \cdot \dfrac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 =$$

2-adic valuation

a and b odd, $x = 2^n \cdot \frac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 = 1/2$$

2-adic valuation

a and b odd, $x = 2^n \cdot \dfrac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 = 1/2$$
$$|4|_2 =$$

2-adic valuation

a and b odd, $x = 2^n \cdot \dfrac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$
\begin{aligned}
|0|_2 &= 0 \\
|1|_2 &= 1 \\
|2|_2 &= 1/2 \\
|6|_2 &= 1/2 \\
|4|_2 &= 1/4
\end{aligned}
$$

2-adic valuation

a and b odd, $x = 2^n \cdot \frac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$
\begin{aligned}
|0|_2 &= 0 \\
|1|_2 &= 1 \\
|2|_2 &= 1/2 \\
|6|_2 &= 1/2 \\
|4|_2 &= 1/4 \\
|20|_2 &=
\end{aligned}
$$

2-adic valuation

a and b odd, $x = 2^n \cdot \dfrac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$
\begin{aligned}
|0|_2 &= 0 \\
|1|_2 &= 1 \\
|2|_2 &= 1/2 \\
|6|_2 &= 1/2 \\
|4|_2 &= 1/4 \\
|20|_2 &= 1/4
\end{aligned}
$$

2-adic valuation

a and b odd, $x = 2^n \cdot \frac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 = 1/2$$
$$|4|_2 = 1/4$$
$$|20|_2 = 1/4$$

$$|1/3|_2 =$$

2-adic valuation

a and b odd, $x = 2^n \cdot \dfrac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 = 1/2$$
$$|4|_2 = 1/4$$
$$|20|_2 = 1/4$$

$$|1/3|_2 = 1$$

2-adic valuation

a and b odd, $x = 2^n \cdot \dfrac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 = 1/2$$
$$|4|_2 = 1/4$$
$$|20|_2 = 1/4$$

$$|1/3|_2 = 1$$
$$|5/3|_2 =$$

2-adic valuation

a and b odd, $x = 2^n \cdot \dfrac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 = 1/2$$
$$|4|_2 = 1/4$$
$$|20|_2 = 1/4$$

$$|1/3|_2 = 1$$
$$|5/3|_2 = 1$$

2-adic valuation

a and b odd, $x = 2^n \cdot \frac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 = 1/2$$
$$|4|_2 = 1/4$$
$$|20|_2 = 1/4$$

$$|1/3|_2 = 1$$
$$|5/3|_2 = 1$$
$$|1/4|_2 =$$

2-adic valuation

a and b odd, $x = 2^n \cdot \frac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 = 1/2$$
$$|4|_2 = 1/4$$
$$|20|_2 = 1/4$$

$$|1/3|_2 = 1$$
$$|5/3|_2 = 1$$
$$|1/4|_2 = 4$$

2-adic valuation

a and b odd, $x = 2^n \cdot \frac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 = 1/2$$
$$|4|_2 = 1/4$$
$$|20|_2 = 1/4$$

$$|1/3|_2 = 1$$
$$|5/3|_2 = 1$$
$$|1/4|_2 = 4$$
$$|1/20|_2 =$$

2-adic valuation

a and b odd, $x = 2^n \cdot \frac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 = 1/2$$
$$|4|_2 = 1/4$$
$$|20|_2 = 1/4$$

$$|1/3|_2 = 1$$
$$|5/3|_2 = 1$$
$$|1/4|_2 = 4$$
$$|1/20|_2 = 4$$

2-adic valuation

a and b odd, $x = 2^n \cdot \dfrac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 = 1/2$$
$$|4|_2 = 1/4$$
$$|20|_2 = 1/4$$

$$|1/3|_2 = 1$$
$$|5/3|_2 = 1$$
$$|1/4|_2 = 4$$
$$|1/20|_2 = 4$$
$$|3/20|_2 =$$

2-adic valuation

a and b odd, $x = 2^n \cdot \dfrac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 = 1/2$$
$$|4|_2 = 1/4$$
$$|20|_2 = 1/4$$

$$|1/3|_2 = 1$$
$$|5/3|_2 = 1$$
$$|1/4|_2 = 4$$
$$|1/20|_2 = 4$$
$$|3/20|_2 = 4$$

2-adic valuation

a and b odd, $x = 2^n \cdot \dfrac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0$$
$$|1|_2 = 1$$
$$|2|_2 = 1/2$$
$$|6|_2 = 1/2$$
$$|4|_2 = 1/4$$
$$|20|_2 = 1/4$$

$$|1/3|_2 = 1$$
$$|5/3|_2 = 1$$
$$|1/4|_2 = 4$$
$$|1/20|_2 = 4$$
$$|3/20|_2 = 4$$
$$|13/16|_2 =$$

2-adic valuation

a and b odd, $x = 2^n \cdot \frac{a}{b} \Rightarrow |x|_2 = (1/2)^n$.

$$|0|_2 = 0 \qquad |1/3|_2 = 1$$
$$|1|_2 = 1 \qquad |5/3|_2 = 1$$
$$|2|_2 = 1/2 \qquad |1/4|_2 = 4$$
$$|6|_2 = 1/2 \qquad |1/20|_2 = 4$$
$$|4|_2 = 1/4 \qquad |3/20|_2 = 4$$
$$|20|_2 = 1/4 \qquad |13/16|_2 = 16$$

“All triangles are isoceles.”

For any valuation,

$$|x+y|_2 \leq |x|_2 + |y|_2 \,.$$

“All triangles are isoceles.”

For any valuation,

$$|x + y|_2 \leq |x|_2 + |y|_2 \,.$$

But for a 2-adic valuation, we actually have

$$|x + y|_2 \leq \max\{|x|_2\,, |y|_2\}\,.$$

“All triangles are isoceles.”

For any valuation,

$$|x + y|_2 \leq |x|_2 + |y|_2 \, .$$

But for a 2-adic valuation, we actually have

$$|x + y|_2 \leq \max \{|x|_2 \, , |y|_2\} \, .$$

And if $|x|_2 \neq |y|_2$,

$$|x + y|_2 = \max \{|x|_2 \, , |y|_2\} \, .$$

Valuations of Irrationals

We can compute $|x|_2$ when x is rational.

Valuations of Irrationals

We can compute $|x|_2$ when x is rational.
But what is $|x|_2$ when x is irrational?

Valuations of Irrationals

We can compute $|x|_2$ when x is rational.
But what is $|x|_2$ when x is irrational?

What is $\left|\sqrt{3}\right|_2$?

Valuations of Irrationals

We can compute $|x|_2$ when x is rational.
But what is $|x|_2$ when x is irrational?

What is $\left|\sqrt{3}\right|_2$?

$$\left|\sqrt{3}\right|_2 \cdot \left|\sqrt{3}\right|_2$$

Valuations of Irrationals

We can compute $|x|_2$ when x is rational.
But what is $|x|_2$ when x is irrational?

What is $\left|\sqrt{3}\right|_2$?

$$\left|\sqrt{3}\right|_2 \cdot \left|\sqrt{3}\right|_2 = \left|\sqrt{3} \cdot \sqrt{3}\right|_2 = |3|_2$$

Valuations of Irrationals

We can compute $|x|_2$ when x is rational.
But what is $|x|_2$ when x is irrational?

What is $\left|\sqrt{3}\right|_2$?

$$\left|\sqrt{3}\right|_2 \cdot \left|\sqrt{3}\right|_2 = \left|\sqrt{3} \cdot \sqrt{3}\right|_2 = |3|_2 = 1,$$

Valuations of Irrationals

We can compute $|x|_2$ when x is rational.
But what is $|x|_2$ when x is irrational?

What is $\left|\sqrt{3}\right|_2$?

$$\left|\sqrt{3}\right|_2 \cdot \left|\sqrt{3}\right|_2 = \left|\sqrt{3} \cdot \sqrt{3}\right|_2 = |3|_2 = 1,$$

we must have $\left|\sqrt{3}\right|_2 = 1$.

Valuations of Irrationals

What about $\left|\sqrt{2}\right|_2$?

Valuations of Irrationals

What about $\left|\sqrt{2}\right|_2$?

$$\left|\sqrt{2}\right|_2 \cdot \left|\sqrt{2}\right|_2 = |2|_2$$

Valuations of Irrationals

What about $\left|\sqrt{2}\right|_2$?

$$\left|\sqrt{2}\right|_2 \cdot \left|\sqrt{2}\right|_2 = |2|_2 = \frac{1}{2},$$

Valuations of Irrationals

What about $\left|\sqrt{2}\right|_2$?

$$\left|\sqrt{2}\right|_2 \cdot \left|\sqrt{2}\right|_2 = |2|_2 = \frac{1}{2},$$

and so

$$\left|\sqrt{2}\right|_2 = \sqrt{\frac{1}{2}}.$$

Valuations of Irrationals

What about other real numbers?

Valuations of Irrationals

What about other real numbers?
What about $|\pi|_2$?

Valuations of Irrationals

What about other real numbers?
What about $|\pi|_2$?

The **axiom of choice** implies that there *exists* a 2-adic valuation defined for all real numbers

Valuations of Irrationals

What about other real numbers?
What about $|\pi|_2$?

The **axiom of choice** implies that there *exists* a 2-adic valuation defined for all real numbers—but we can't write an example down.

2-adic valuation: Review

Write $|x|_2$ for the 2-adic valuation of x.

2-adic valuation: Review

Write $|x|_2$ for the 2-adic valuation of x.

$|x|_2$ measures how many times 2 divides x.

2-adic valuation: Review

Write $|x|_2$ for the 2-adic valuation of x.

$|x|_2$ measures how many times 2 divides x.
$|x|_2$ measures x's size from 2's perspective.

2-adic valuation: Review

Write $|x|_2$ for the 2-adic valuation of x.

$|x|_2$ measures how many times 2 divides x.
$|x|_2$ measures x's size from 2's perspective.

Key Observation

If n is an even integer, $|n|_2 < 1$.
If n is an odd integer, $|n|_2 \geq 1$.

The Proof

Two ingredients:

Sperner's Lemma

and a

2-adic valuation.

The Proof

Two ingredients:

Sperner's Lemma

and a

2-adic valuation.

The Proof

Two ingredients:

Sperner's Lemma

and a

2-adic valuation.

The Original Question

We want to show that a square cannot be cut into an odd-number of equal-area triangles.

The Original Question

We want to show that a square cannot be cut into an odd-number of equal-area triangles.

So far, we have

- Sperner's lemma, and
- a 2-adic valuation.

How can we use these tools?

The Original Question

We want to show that a square cannot be cut into an odd-number of equal-area triangles.

So far, we have

- Sperner's lemma, and
- a 2-adic valuation.

How can we use these tools?

Paint by number. Use the 2-adic valuation to decide what color to give to the vertices.

The Proof

Given: a **triangulation** of a square into *n* equal-area triangles.

The Proof

Given: a **triangulation** of a square into n equal-area triangles.

Color a vertex at (x, y) with

- **A** if $|x|_2 < 1$ and $|y|_2 < 1$.
- **B** if $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.
- **C** if $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

The Proof

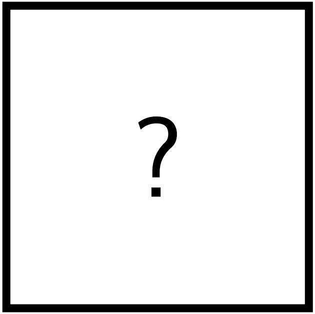
Given: a **triangulation** of a square into n equal-area triangles.

Color a vertex at (x, y) with

- **A** if $|x|_2 < 1$ and $|y|_2 < 1$.
- **B** if $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.
- **C** if $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

We want to **prove** that n is even.

Use Sperner's Lemma

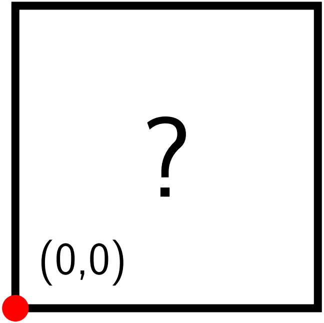


A: $|x|_2 < 1$ and $|y|_2 < 1$.

B: $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.

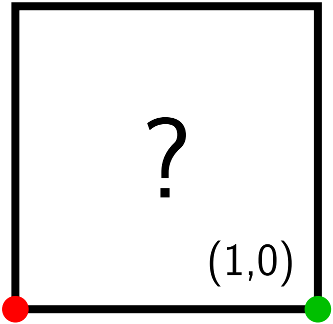
C: $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

Use Sperner's Lemma



A: $|x|_2 < 1$ and $|y|_2 < 1$.
B: $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.
C: $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

Use Sperner's Lemma

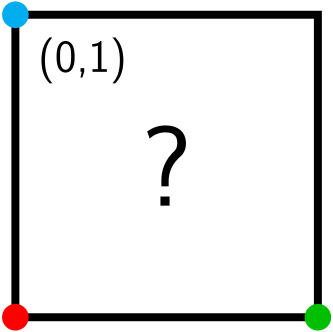


A: $|x|_2 < 1$ and $|y|_2 < 1$.

B: $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.

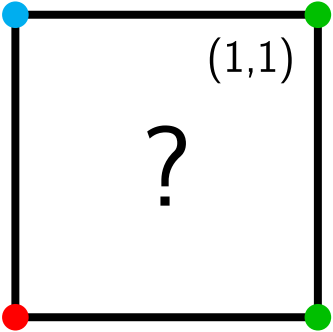
C: $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

Use Sperner's Lemma



A: $|x|_2 < 1$ and $|y|_2 < 1$.
B: $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.
C: $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

Use Sperner's Lemma

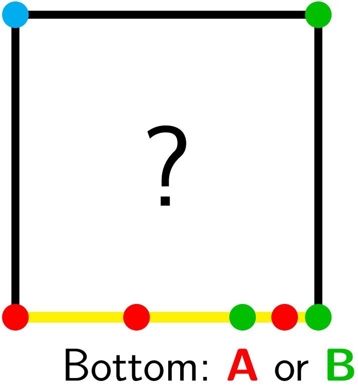


A: $|x|_2 < 1$ and $|y|_2 < 1$.

B: $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.

C: $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

Use Sperner's Lemma

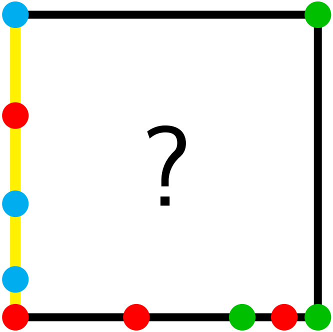


A: $|x|_2 < 1$ and $|y|_2 < 1$.

B: $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.

C: $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

Use Sperner's Lemma



A: $|x|_2 < 1$ and $|y|_2 < 1$.

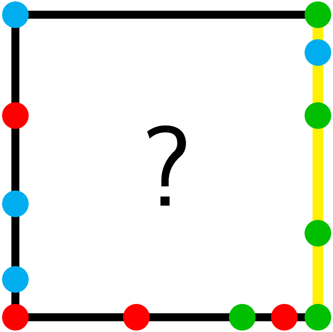
B: $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.

C: $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

Bottom: **A** or **B**

Left: **A** or **C**

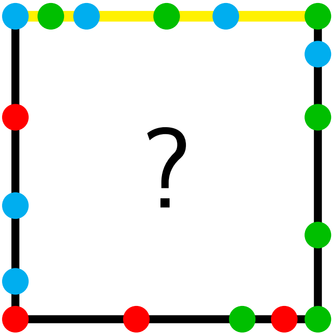
Use Sperner's Lemma



A: $|x|_2 < 1$ and $|y|_2 < 1$.
B: $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.
C: $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

Bottom: **A** or **B**
Left: **A** or **C**
Right: **B** or **C**

Use Sperner's Lemma



A: $|x|_2 < 1$ and $|y|_2 < 1$.

B: $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.

C: $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

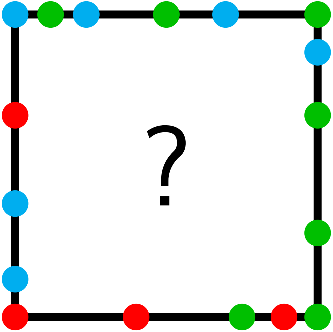
Bottom: **A** or **B**

Left: **A** or **C**

Right: **B** or **C**

Top: **B** or **C**

Use Sperner's Lemma

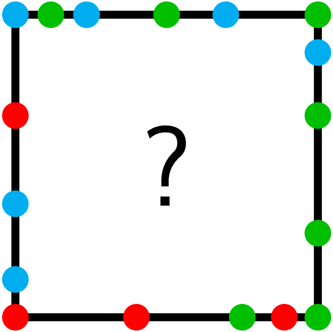


A: $|x|_2 < 1$ and $|y|_2 < 1$.
B: $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.
C: $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

Bottom: **A** or **B**
Left: **A** or **C**
Right: **B** or **C**
Top: **B** or **C**

$\Rightarrow$ Odd number of **AB** edges on perimeter

Use Sperner's Lemma

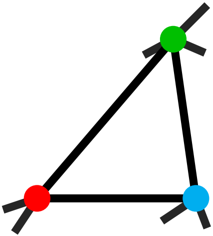


A: $|x|_2 < 1$ and $|y|_2 < 1$.
B: $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.
C: $|y|_2 \geq 1$ and $|x|_2 < |y|_2$.

Bottom: **A** or **B**
Left: **A** or **C**
Right: **B** or **C**
Top: **B** or **C**

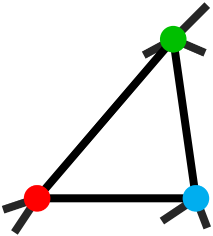
$\Rightarrow$ Odd number of **AB** edges on perimeter; get **ABC** triangle

What we know so far



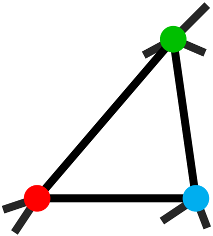
In any triangulation of the square into n triangles, there is an **ABC** triangle.

What we know so far



In any triangulation of the square into *n* equal-area triangles, there is an **ABC** triangle.

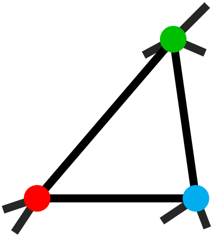
What we know so far



In any triangulation of the square into n equal-area triangles, there is an **ABC** triangle.

Area of **ABC** triangle is $\frac{1}{n}$.

What we know so far

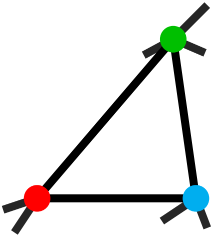


In any triangulation of the square into n equal-area triangles, there is an **ABC** triangle.

Area of **ABC** triangle is $\frac{1}{n}$.

Key idea: the area of a triangle is related to the color of its vertices.

What we know so far



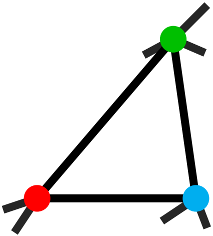
In any triangulation of the square into n equal-area triangles, there is an **ABC** triangle.

Area of **ABC** triangle is $\frac{1}{n}$.

Key idea: the area of a triangle is related to the color of its vertices.

In fact, if r is the area of an **ABC** triangle, then $|r|_2 \geq 2$.

What we know so far



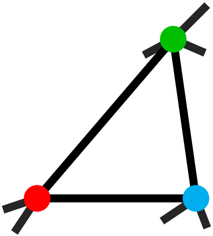
In any triangulation of the square into n equal-area triangles, there is an **ABC** triangle.

Area of **ABC** triangle is $\frac{1}{n}$.

Key idea: the area of a triangle is related to the color of its vertices.

In fact, if r is the area of an **ABC** triangle, then $|r|_2 \geq 2$. So $|1/n|_2 \geq 2$.

What we know so far



In any triangulation of the square into n equal-area triangles, there is an **ABC** triangle.

Area of **ABC** triangle is $\frac{1}{n}$.

Key idea: the area of a triangle is related to the color of its vertices.

In fact, if r is the area of an **ABC** triangle, then $|r|_2 \geq 2$. So $|1/n|_2 \geq 2$. So n is even.

Area and vertex color

$r =$ area of **ABC** triangle with vertices

$(\ 0,\ 0\ )$ colored **A**

$(x_b,\ y_b)$ colored **B** so $|x_b|_2 \geq 1$ and

$$|x_b|_2 \geq |y_b|_2$$

$(x_c,\ y_c)$ colored **C** so $|y_c|_2 \geq 1$ and

$$|y_c|_2 > |x_c|_2$$

Area and vertex color

$r =$ area of **ABC** triangle with vertices

(0, 0) colored **A**

(x_b, y_b) colored **B** so $|x_b|_2 \geq 1$ and

$|x_b|_2 \geq |y_b|_2$

(x_c, y_c) colored **C** so $|y_c|_2 \geq 1$ and

$|y_c|_2 > |x_c|_2$

$$r = \frac{1}{2} \cdot (x_b\, y_c - x_c\, y_b)$$

Area and vertex color

$r =$ area of **ABC** triangle with vertices

$(\ 0,\ 0\ )$ colored **A**

$(x_b,\ y_b)$ colored **B** so $|x_b|_2 \geq 1$ and

$|x_b|_2 \geq |y_b|_2$

$(x_c,\ y_c)$ colored **C** so $|y_c|_2 \geq 1$ and

$|y_c|_2 > |x_c|_2$

$$|r|_2 = \left|\frac{1}{2}\right|_2 \cdot |x_b\, y_c - x_c\, y_b|_2$$

Area and vertex color

$r =$ area of **ABC** triangle with vertices

$(\ 0,\ 0\ )$ colored **A**

$(x_b,\ y_b)$ colored **B** so $|x_b|_2 \geq 1$ and

$|x_b|_2 \geq |y_b|_2$

$(x_c,\ y_c)$ colored **C** so $|y_c|_2 \geq 1$ and

$|y_c|_2 > |x_c|_2$

$$|r|_2 = \quad 2 \quad \cdot |x_b\, y_c - x_c\, y_b|_2$$

Area and vertex color

$r =$ area of **ABC** triangle with vertices

$(\ 0,\ 0\ )$ colored **A**

$(x_b,\ y_b)$ colored **B** so $|x_b|_2 \geq 1$ and $|x_b|_2 \geq |y_b|_2$

$(x_c,\ y_c)$ colored **C** so $|y_c|_2 \geq 1$ and $|y_c|_2 > |x_c|_2$

$$|r|_2 = \quad 2 \quad \cdot |x_b\, y_c - x_c\, y_b|_2$$

$$= 2 \cdot \max\{|x_b y_c|_2\,, |x_c y_b|_2\}$$

Area and vertex color

$r =$ area of **ABC** triangle with vertices

$(\ 0,\ 0\ )$ colored **A**

$(x_b,\ y_b)$ colored **B** so $|x_b|_2 \geq 1$ and $|x_b|_2 \geq |y_b|_2$

$(x_c,\ y_c)$ colored **C** so $|y_c|_2 \geq 1$ and $|y_c|_2 > |x_c|_2$

$$|r|_2 = \quad 2 \quad \cdot |x_b\, y_c - x_c\, y_b|_2$$

$$= 2 \cdot \max\{|x_b y_c|_2\,, |x_c y_b|_2\}$$

$$= 2 \cdot |x_b y_c|_2$$

Area and vertex color

$r =$ area of **ABC** triangle with vertices

$(\ 0,\ 0\ )$ colored **A**

$(x_b,\ y_b)$ colored **B** so $|x_b|_2 \geq 1$ and

$|x_b|_2 \geq |y_b|_2$

$(x_c,\ y_c)$ colored **C** so $|y_c|_2 \geq 1$ and

$|y_c|_2 > |x_c|_2$

$$|r|_2 = \quad 2 \quad \cdot |x_b\, y_c - x_c\, y_b|_2$$

$$= 2 \cdot \max\{|x_b y_c|_2\,, |x_c y_b|_2\}$$

$$= 2 \cdot |x_b|_2 \cdot |y_c|_2$$

Area and vertex color

$r =$ area of **ABC** triangle with vertices

$(\ 0,\ 0\ )$ colored **A**

$(x_b,\ y_b)$ colored **B** so $|x_b|_2 \geq 1$ and

$|x_b|_2 \geq |y_b|_2$

$(x_c,\ y_c)$ colored **C** so $|y_c|_2 \geq 1$ and

$|y_c|_2 > |x_c|_2$

$$|r|_2 = \quad 2 \quad \cdot |x_b\, y_c - x_c\, y_b|_2$$

$$= 2 \cdot \max\{|x_b y_c|_2\,, |x_c y_b|_2\}$$

$$= 2 \cdot |x_b|_2 \cdot |y_c|_2 \geq 2.$$

Area and vertex color

Let r be the area of an **ABC** triangle, with vertices

$(x_a \quad , y_a \quad)$ colored **A**

$(x_b \quad , y_b \quad)$ colored **B**

$(x_c \quad , y_c \quad)$ colored **C**

Area and vertex color

Let r be the area of an **ABC** triangle, with vertices translated by $(-x_a, -y_a)$

$$(x_a - x_a, y_a - y_a) \text{ colored } \mathbf{A}?$$
$$(x_b - x_a, y_b - y_a) \text{ colored } \mathbf{B}?$$
$$(x_c - x_a, y_c - y_a) \text{ colored } \mathbf{C}?$$

Area and vertex color

Let r be the area of an **ABC** triangle, with vertices translated by $(-x_a, -y_a)$

$(x_a - x_a, y_a - y_a)$ colored **A**
$(x_b - x_a, y_b - y_a)$ colored **B**
$(x_c - x_a, y_c - y_a)$ colored **C**

Translating by a point colored **A** preserves the colors.

Area and vertex color

Let r be the area of an **ABC** triangle, with vertices translated by $(-x_a, -y_a)$

$(x_a - x_a, y_a - y_a)$ colored **A**
$(x_b - x_a, y_b - y_a)$ colored **B**
$(x_c - x_a, y_c - y_a)$ colored **C**

Translating by a point colored **A** preserves the colors.
The previous calculation proves $|r|_2 \geq 2$.

Area and vertex color

Let n be the number of equal-area triangles, each having area $r = 1/n$.

Area and vertex color

Let n be the number of equal-area triangles, each having area $r = 1/n$.

Then $|n \cdot r|_2 = |1|_2 = 1$.

Area and vertex color

Let n be the number of equal-area triangles, each having area $r = 1/n$.

Then $|n \cdot r|_2 = |1|_2 = 1$.

But $|r|_2 \geq 2$, so $|n|_2 \leq 1/2$, so n is even.

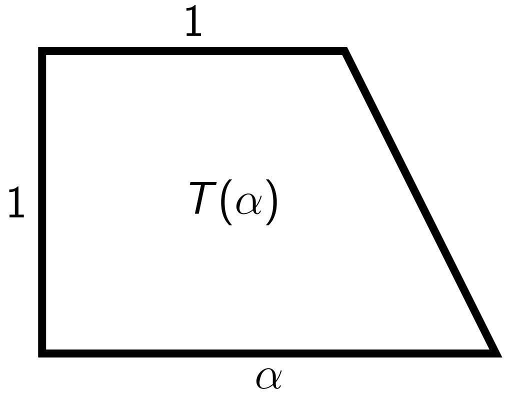
Area and vertex color

Let n be the number of equal-area triangles, each having area $r = 1/n$.

Then $|n \cdot r|_2 = |1|_2 = 1$.

But $|r|_2 \geq 2$, so $|n|_2 \leq 1/2$, so n is even. $\square$

Other polygons



n is in the **spectrum** of a polygon P
if P can be divided in n equal-area triangles.

Bibliography

Where can I learn more?

Bibliography

Where can I learn more?

- Paul Monsky. “On dividing a square into triangles.” *Amer. Math. Monthly* **77** 1970 161–164.

Bibliography

Where can I learn more?

- Paul Monsky. “On dividing a square into triangles.” *Amer. Math. Monthly* **77** 1970 161–164.
- Sherman Stein and Szabó Sándor. *Algebra and tiling. Homomorphisms in the service of geometry.* Carus Mathematical Monographs **25**. MAA. 1994.

Thank You!