## How to impress your friends thanks to the Fibonacci sequence. . .

Now, here is a magic trick you can play with friends.
First of all, write 1.62 on a sheet of paper your friend can't see. Ask him to choose two integers $a$ and $b$, for instance between 1 and 20, and to write them on the first two lines of the following table $(a>0, b>0)$ :

| $u(1)$ | $\mathbf{a}$ |  |
| :---: | :---: | :---: |
| $u(2)$ | $\mathbf{b}$ |  |
| $u(3)$ | $\mathbf{a}+\mathbf{b}$ |  |
| $u(4)$ | $\mathbf{a}+\mathbf{2 b}$ |  |
| $u(5)$ | $\mathbf{2 a}+\mathbf{3 b}$ |  |
| $u(6)$ | $\mathbf{3 a}+\mathbf{5 b}$ |  |
| $u(7)$ | $\mathbf{5 a}+\mathbf{8 b}$ |  |
| $u(8)$ | $\mathbf{8 a}+\mathbf{1 3 b}$ |  |
| $u(9)$ | $\mathbf{1 3 a}+\mathbf{2 1 b}$ |  |
| $u(10)$ | $\mathbf{2 1 a}+\mathbf{3 4 b}$ |  |

Then, he has to fulfill the table, such that each line is the the sum of both preceding lines.

Now, ask him to sum all the lines of the table. You can give him the result very quickly. Then, tell him you had predicted the ratio $R$ of the tenth and the nineth number. Show him the number you had written on the sheet of paper at the beginning.

How is this possible?

Fullfill the table when the chosen integers are $a$ and $b$ and compute the sum $S$. What do you notice?

$$
\begin{aligned}
& \mathrm{S}=\mathrm{a}+\mathrm{b}+(\mathrm{a}+\mathrm{b})+(\mathrm{a}+2 \mathrm{~b})+(2 \mathrm{a}+3 \mathrm{~b})+(3 \mathrm{a}+5 \mathrm{~b}) \\
& \quad \quad+(5 \mathrm{a}+8 \mathrm{~b})+(8 \mathrm{a}+13 \mathrm{~b})+(13 \mathrm{a}+21 \mathrm{~b})+(21 \mathrm{a}+34 \mathrm{~b}) \\
& \quad \mathrm{S}=\mathbf{a}(1+1+1+2+3+5+8+13+21)+\mathbf{b}(1+1+2+3+5+8+13+21+34) \\
& \mathrm{S}=55 \mathrm{a}+88 \mathrm{~b} \\
& \mathrm{~S}=11.5 \mathrm{a}+11.8 \mathrm{~b} \\
& \mathrm{~S}=11 .(5 \mathrm{a}+8 \mathrm{~b}) \\
& \mathrm{S}=11 . \mathbf{u}(7)
\end{aligned}
$$

To obtain the sum of the ten numbers, you must multiply the seventh term by 11.

Theorem. If $A, B, C$ and $D$ are positive numbers such that $\frac{A}{C} \leq \frac{B}{D}$, then:

$$
\frac{A}{C} \leq \frac{A+B}{C+D} \leq \frac{B}{D}
$$

Use this theorem to show that $1.615 \leq R \leq 1.62$ and conclude.
$u(10)=21 a+34 b$ and $u(9)=13 a+21 b$ so that $R=\frac{21 a+34 b}{13 a+21 b}$.
Let $\mathrm{A}=21 \mathrm{a}, \mathrm{B}=34 \mathrm{~b}, \mathrm{C}=13 \mathrm{a}, \mathrm{D}=21 \mathrm{~b}$.
$\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ are positive numbers $(\mathrm{a}>0, \mathrm{~b}>0)$ and: $\frac{\mathrm{A}}{\mathrm{C}}=\frac{21 \mathrm{a}}{13 \mathrm{a}}=\frac{21}{13}, \frac{\mathrm{~B}}{\mathrm{D}}=\frac{34 \mathrm{~b}}{21 \mathrm{~b}}=\frac{34}{21}$.
We have:
$13.34-21.21=442-441=1$, so that $13.34>21.21$, that is
so that $\frac{34}{21}>\frac{21}{13}$. We deduce: $\frac{\mathrm{A}}{\mathrm{C}} \leq \frac{\mathrm{B}}{\mathrm{D}}$.
Then, we can apply the Theorem. We have: $\frac{A}{C} \leq \frac{A+B}{C+D} \leq \frac{B}{D}$, that is $\frac{21}{13} \leq R \leq \frac{34}{21}$.
As $\frac{21}{13} \simeq 1.6154$ and $\frac{34}{21} \simeq 1.619$, we have $1.615 \leq \frac{21}{13}$ and $\frac{34}{21} \leq 1.62$.
We deduce $1.615 \leq \frac{\mathrm{A}}{\mathrm{C}} \leq \mathrm{R} \leq \frac{\mathrm{B}}{\mathrm{D}} \leq 1.62$, so that $1.615 \leq \mathrm{R} \leq 1.62$ :
the ratio R is 1.62 to 2 decimal places.

Prove the theorem: notice that, since the numbers are all positive, the inequality $\frac{A}{C} \leq \frac{A+B}{C+D}$ is equivalent to $A .(C+D) \leq C .(A+B)$.

We have:
$\mathbf{A} \cdot(\mathbf{C}+\mathbf{D})-\mathbf{C} \cdot(\mathbf{A}+\mathbf{B})=\mathbf{A C}+\mathbf{A D}-(\mathbf{C A}+\mathbf{C B})=\mathbf{A C}+\mathbf{A D}-\mathbf{C A}-\mathbf{C B}=\mathbf{A D}-\mathbf{B C}$.
As $A, B, C, D$ are positive numbers such that $\frac{A}{C} \leq \frac{B}{D}$, this means $A D \leq B C$. We deduce:
A. $(\mathbf{C}+\mathbf{D})-\mathbf{C} \cdot(\mathbf{A}+\mathrm{B}) \leq \mathbf{0}$, that is $\mathbf{A} .(\mathbf{C}+\mathbf{D}) \leq \mathrm{C} .(\mathrm{A}+\mathrm{B})$.

As $\mathrm{C}+\mathrm{D}>0$, we can divide both members of the inequality by $\mathrm{C}+\mathrm{D}$, we obtain an inequality in the same order:
$\frac{\mathbf{A} \cdot(\mathbf{C}+\mathbf{D})}{\mathrm{C}+\mathrm{D}} \leq \frac{\mathrm{C} \cdot(\mathrm{A}+\mathrm{B})}{\mathrm{C}+\mathrm{D}}$, that is $\mathrm{A} \leq \frac{\mathrm{C} \cdot(\mathrm{A}+\mathrm{B})}{\mathrm{C}+\mathrm{D}}$.
Now, as $\mathrm{A}>0$, we can divide both members of the inequality by A , we obtain an inequality in the same order:
$\frac{\mathrm{A}}{\mathrm{C}} \leq \frac{\mathrm{A}+\mathrm{B}}{\mathrm{C}+\mathrm{D}}$.
Considering B. $(\mathbf{C}+\mathbf{D})-\mathrm{D} .(\mathrm{A}+\mathrm{B})$, an analogue proof shows that $\frac{\mathrm{A}+\mathrm{B}}{\mathrm{C}+\mathrm{D}} \leq \frac{\mathrm{B}}{\mathrm{D}}$.
References: IREM de La Réunion; APMEP; Su, Francis E., et al. "Leapfrog Addition." Math Fun Facts. [http://www.math.hmc.edu/funfacts](http://www.math.hmc.edu/funfacts).

